Journal des Sciences
I.S.S.N 0851 – 4631

MONIC POLYNOMIALS GENERATING SEQUENCES ON SEMIFIELDS

Lamine Ngom,
Department of Mathematics and computer sciences,
Cheikh Anta Diop University, Dakar – Senegal,
e-mail : laminengo89@hotmail.fr

Cherif Bachir DEME
Department of TIC,
Alioune Diop University, Bambey– Senegal
Auteur correspondant : Ousseynou.diop@ucad.edu.sn

Oumar DIANKHA, *e-mail:oumar.diankha@ucad.edu.sn*
Department of Mathematics and computer sciences,
Cheikh Anta Diop University, Dakar – Senegal

Résumé. Un polynôme unitaire$G(x) = x^m - g_{m-1}x^{m-1} - \cdots - g_0 \in S[x]$génère un segment $\overline{u(0,l)}$d'une séquence u de longueur lsi l ≤ m ou l ≥ m et$u(m+n) = g_{m-1}u(m+n-1) - \ldots - g_0u(n)$ pour$n \in \overline{0,l-m+1}$. Dans ce papier, nous étudions quelques propriétés des polynômes unitaires de degré minimal générant tout segment d'une séquence à valeurs dans un semi corps zero-sum-free.

Abstract. A monic polynomial $G(x) = x^m - g_{m-1}x^{m-1} - \cdots - g_0 \in S[x]$generates a segment $\overline{u(0,l)}$of sequence u of length l of the sequence u if either l ≤ m or l ≥ m and $u(m+n) = g_{m-1}u(m+n-1) - . . . . - g_0u(n)$ for $n \in \overline{0,l-m+1}$. In this paper, we study some properties of monic polynomials of minimal degree which generates any segment of sequences with values in a zero-sum-free halffields.

Introduction

Linear recurrings sequences over zero-sum semirings was first introduced in [3]. In this paper,we study some properties of monic polynomials of minimal degree which generates any segment ofsequences with values in a zero-sum-free halffields S.A monic polynomial $G(x) = x^m - g_{m-1}x^{m-1} - \dots \dots \; g_0 \in S[x]$generates a segment $u(0,l)$ of sequence u of length l of the sequence u if eitherl $\leq$ m or l $\geq$ m and $u(m+n) = g_{m-1}u(m+n-1) - \cdots - g_0 u(n)$for $n \in \overline{0, l-m+1}$. Note thatthese polynomials will not be in S[x], but also in D(S)[x]. After some preliminaries on halffields andlinear recurring sequences, we characterize in section 2 monic polynomials generating any segment of sequences with values in halffields. We have proved that for any linear recurring sequence u, ifF and G are comaximales, then there exists an extension v of u with the help of G such that FGgenerates any segment of v. We have also given in this section a distinct new improvement of theBerlekamp-Massey algorithm on linear recurring sequences over zero-sum-free halffields.

1. Preliminaries

A semifield is a commutative semiring with identity such that each non-zero element of S isinvertible. It is said to be zero-sum-free if it has no zero-sum elements different from zero. Recall that any Cancellative semifield (brirfly halfield) is embedable in a ring (equivalently, embedable in his ring of difference).

In [1], it is shown that every semifield is either a field or zero-sum-free semifield, in [7], theauthors prove that if S a additively cancellative semifield, then D(S) is a field.

Consequently, if S a additively cancellative semifield, then D(S)[x] is a strongly euclidian ring.

Lemma 1.1. If S be a halfield (additively cancelative semifield), then S[x]is additively cancelativeand D(S[x]) = D(S)[x].

Proof. Let $F = (a_0, a_1, \dots, a_k, \dots)$,$G = (b_0, b_1, \dots, b_k, \dots)andH = (c_0, c_1, \dots, c_k, \dots)$such that$F + H = F + G$.This equality implies that:$a_0 + b_0 = a_0 + c_0, a_1 + b_1 = a_1 + c_1, \dots, a_k + b_k = a_k + c_k, \dots$.Since S is additivelycancelative, then $b_0 = c_0, b_1 = c_1, \dots, b_k =$

c_k ....Hence $F = G$ and S[x] is additively cancelative.Since $H = F - G$where$F = (a_0, a_1, \dots, a_k \dots)$; $G = (b_0, b_1, \dots, b_k \dots) \in$ S[x], then $H = (a_0 - b_0, a_0 - b_1, \dots, a_k - b_k \dots)$.

The Berlekamp-Massey algorithm built for a given sequence u and$l \in N$ a polynomial of minimaldegree which generates $\overline{u(0, l-1)}$. For an arbitrary monic polynomial $G(x) \in S[x]$, let $K_u(G)$ bethe number of zero at the begining of the sequence $v = G(x)u$and let $l_u(G) = K_u(G) + \deg(G)$.

Then $l_u(G)$ is the maximum length of the segment $u\overline{(0, l-1)}$generated by $G(x)$. The rank ofa segment $u\overline{(0, -1)}$of a sequence u is the minimum $m_u(l)$ of the degrees of monic polynomials which are generated of$\overline{u(0, l-1)}$. The Berlekamp-Massey algorithm builds a monic polynomial$G(x) \in S[x]$ with the property $L_u(G) \geq$ l and$\deg\big(G(x)\big) = m_u(l)$. This version of the algorithm is as follows. We will build a sequence $G_0, G_1, G_2, \dots, G_t$ ...as polynomials of the degrees $m_0 \leq m_1 \leq m_2 \dots \leq \cdots$ according to the next rule.

(1) We put on $G_0 = 1, M_0 = 0$ and $u_0\overline{(0, l - m_0 - 1)} = u_0\overline{(0, l-1)}$where $u_0 = G_0 u = u$.If $u_0(0, l-1) = 0$, then $l_u(G_0) \geq l$ and $G(x) = G_0(x) m_u(l) = m_0 = 0$. Otherwise $u_0 = (0, \dots, u_0(k_0), \dots), u_0(k_0) \neq 0, k_0 = k_0(G) \leq l - m_0$.

(2) We put on $G_1(x) = x^{k_0+1} - u_0(k_0+1) u_0(k_0)^{-1} x^{k_0} G_0(x), m_1 = k_0 + 1$ and calculate$u_1\overline{(0, l - m_1 - 1)}$ where $u_1 = G_1 u$. If $u_1\overline{(0, l - m_1 - 1)} = 0$, then $l_u(G_1) \geq l$ and $G(x) = G_1(x), m_u(l) = m_1 = 0$. Otherwise $u_1 = (0, \dots, u_1(k_1), \dots), u_1(k_1) \neq 0, k_1 = k_1(G) \leq l - m_1$.

(3) Now let us suppose that we have built the polynomial $G_0, G_1, \dots, G_t$ and let $k_j = k_u G_j$, $l_j = l_u\big(G_j\big) = k_j + m_j$ for $j \in 0, t$.

(4) We will define $s = \max\big\{ i \in 0, t \; m_j < m_t \big\}$ and put

$$G_{t+1}(x) = \begin{cases} G_t(x) - x^{(k_s)-(k_t)} u_t(k_t) u_s\big(k_s\big)^{-1} G_s(x) \;\; if \;\; k_t \leq k_s\big) \\ x^{k_t - k_s} G_t(x) - u_t(k_t) u_s\big(k_s\big)^{-1} G_s(x) \;\; if \;\; k_t > k_s\big) \end{cases}$$

Where$u_j G_t u$, for $j \in 0, t$ .Let $u_{t+1} = G_{t+1} u$. Than $l_u(G_{t+1}) > l_u(G_t)$ and the following statement is true: either $u_{t+1}\overline{(0, l - m_{t+1} - 1)} = 0$ and $G_{(t+1)(x)} = G(x), m_u(l) = m_{t+1}$

or $l_u(G_{t+1}(x)) = k_{t+1} + m_{t+1} = l_{t+1} < l$ and $G_{t+1}(x)$ is the polynomial of minimal degree which generates the sequence $u\overline{(0, l_{t+!} - 1)}$.

2. **MONIC POLYNOMIALS GENERATING SEQUENCES ON SEMIFIELDS**

Recall that the sequence u is said linear recurring of order k over semifield S if there are $a_0, a_1, \ldots, a_{k-1}$ in S such that $u(n+k) = a_{(k-1)u(n+k-1)} + \ldots + a_0 u(n)$.

Since the considered halffied are zero-sum-free, then the characteristic polynomial F can not belong to $S[x]$ but have his coefficients in the ring of difference of $S[x]$. The minimal polynomial of u is the characteristic polynomial of u of minimal degree. In [3] it is proved that a sequence is a linear recurring sequence if $An_{S[x]}(x)$ is a monic ideal (that is an ideal containing monic polynomial). A minimal polynomial of linear recurring sequence over halffield is monic polynomial which generated any segment of u.

Proposition 2.1. Let u be linear recurring over S. If F is minimal polynomial of u, then F generates any segment u ;

Proof: Assume that$u$ is linear recurring sequence over S of characteristic polynomial $F = x^l - a_{l-1}x^{l-1} - \cdots - a_0$. Since $S \in An_{D(x)[s]}$, then $u(n+l) = a_{l-1}u(n+l-1) + \ldots + a_0 u(n)$ for all positive integer n . Hence F generates any segment $\overline{u(0, k-1)}$ with $k \geq l$.

Proposition 2.2. Let u be a linear recurring sequence over S . Then the minimal polynomial of u is unique if and only if there exist a monic polynomial F such that $An_{D(s)[x]}(u) = D(S)[x]F$.

Proof: Let F be the minimal polynomial. Since F is unique, then for all $G \in An_{D(s)[x]}(u), F divide G$. Consequently $An_{D(s)[x]}(u) \ll D(S)[x]F$. Since $D(S)[x]F \ll An_{D(s)x[x]}(u)$, then $An_{D(s)[x]}(u) = D(S)[x]F$. Then $An_{D(s)[x]}(u) = D(S)[x]F$. Conversely if $An_{D(s)[x]}(u) = D(S)[x]F$, then $F$is a minimal polynomial. If G is anather minimal polynomial,$F divide G$ and $\deg(F) = \deg(G)$ then $F = G$.

Corollary 2.3. If u is a linear recurring sequence over a additively cancelative semifield S, then the minimal polynomial of u is uniqu .

Proof: If S is a cancelative semifield, then $D(S)[x]$ is a strongly euclidian semiring. Therefor $An_{D(s)[x]}(u)$ is a principal ideal of $D(S)\ [x]F$. Hence $F$is the unique minimal polynomial of u.

Proposition2.4. Let$u$ a sequence in S , F an irreductible monic polynomial with coefficient in S. If F generate any segment of u,then u is linear recurring sequence of minimal polynomial F.

Proof: Let u be a sequence in S and $F = x^k + a_{k-1}x^{k-1} + \cdots + a_0$ where $a_i \in \overline{0\,,\mathrm{k}-1}$ are zero-sum. Since F generate any segment of u , then for all positive integer n, there existe $b_i\,, i \in \overline{0,\mathrm{k}-1}$ such that$a_i + b_i = 0$ and $u(i+k) = b_{(k-1)u(i+k-1)} + \cdots + b_0u(i)$. Then u is linear recurring sequence over S .

By using preceding results, we have the next characterization of monic polynomial F generating of sequence with values in a zero-sum-free halffield.

Proposition 2.5. Let S a zero-sum-free halffield and u a sequence with values in S. Then the following statements are equivalents:

(1) F generates any segment of u;

(2) F is the the unique minimal polynomial of u;

(3) $An_{D(s)[x]}(u) = D(S)[x]F$;

(4) u is linear recurring sequence over S of minimal polynomial F;

Proof: (4)=> (3) and (2) => (3) follow from proposition 2.2, (2) => (1) from proposition 2.1, (1) => (4) from position 2.4.

The followings two propositions shows that the set of monic polynomial generating any segment of a linear recurring sequence over S is an $S[x]$-semimodule.

Proposition 2.6. Let F be an irreducible monic polynomial generating any segment of u. If G is a monic polynomial, FG generates any segment of u.

Proof: If F generated any segment of u, then by proposition 2.4 F is minimal polynomial. Therefore $FG \in An_{D(s)[x]}(u)$. Hence FG generates any segments of u.

Proposition 2.7. Let u a sequence in S and $l \in N^*$. If F and G are monic polynomials generating the segment$\overline{u(\ 0\ l-1)}$, then$F+G$ generates $\overline{u(\ 0\ ,l-1)}$.

Proof: Let $F = x^h + \sum_{i=0}^{h-1} a_i x^i$, $G = x^k + \sum_{j=0}^{k-1} b_j x^j \in D(S)[x]$ generating the segment$\overline{u(0\ ,l-1)}$ (where $h \geq k$). Then for all $n \in \overline{0, l-h-1}$, $(F+G)u(n) = (x^h + \sum_{i=0}^{k-1} a_i x^i + (1+b_k)x^k + \sum_{j=k-1}^{h-1}(a_j+b_j)x^j) = u(n+h) + \sum_{i=0}^{k+1} a_i u(n+i) +$
$(1+b_k)u(n+k) + \sum_{j=k-1}^{h-1}(a_j+b_j)u(n+j) = u(n+h) + \sum_{i=0}^{h-1} a_i u(n+i) +$
$u(n+k) + \sum_{j=0}^{k-1} b_j u(n+j) = 0$. Therefore $F+G$generates the segment$\overline{u(0, l-1)}$.

Let u be with values in a semifield,$F = x^m - \sum_{i=0}^{m-1} a_i x^i$ be a monic polynomial and $\alpha \in S^m$ the sequence v define by:

$$v(0,m) = \propto \text{ and } v(m+n) = \sum_{i=0}^{m-1} a_i v(i+n+u(n))$$

is call an extension of the sequence u with the help of the polynomial $F$and the vector $\propto$. We denote by $v = (\propto, u, F)$.

Remark 2.1. Every linear recurring sequence is an extension of the sequence with the help of his minimal polynomial and his initial vector.

Let F be the monic polynomial of degree k and A an ideal of $S[x]$. Recall that e^F is the periodic linear recurring sequence over S with minimal polynomial F with initial vector is $(0, \dots, 1)$ and $(A{:}\,F) = \{G \in S[x], FG \in A\}$.

Proposition 2.8 Let u be withh values in S, F and G be monic polynomials such that $\deg(F) = m$, $\propto \in s^m$ and $v = (\propto, u, F)$. Then G generate any segments of u if and only if FG generate any segment of v.

Proof: Suppose G generates any segment of u. Since $v = (\propto, u, F)$, then for all $n, \in N$, $(FG)v(n) = FGv(n) = gu(n) = 0$. Therefore FG generates any segment of v. conversaly, if FG generate any segment of v, then for all $n \in N$, $Gu(n) = GFv(n) = 0$. Hence G generate any segment of u.

Corollary 2.9. If u be with values in S, F be a monic polynomial of $m, \propto \in S^m$ and $v = (\propto, u, F)$ then $An_{D(S)[x]}(u) = An_{D(S)[x]}(v) : F$.

Let us recall a well know result about fields, which can be found for instance in [5].

Lemma 2.10. Let S be a field I_1 and I_2 two monic comaximal ideals from $S[x]$. If $u_i \in L_S(I_i), i = 1, 2$ then $An_{S[x]}(u_1 + u_2) = An_{S[x]}(u_1) An_{S[x]}(u_2)$.

Proposition 2.11. Let u be a linear recurring sequence of minimal polynomial F and G a monic polynomial. If F and G are comaximales, then there exists an extension v of u with the help of G such that FG generate any segment of v.

Proof: Since F and G are comaximales, there exist $B, C \in D(S)[x]$ such that $GB + FC = 1$ then $BGu + FCu = u$. Since $Fu = 0$, then $GBu = u$. Therefore $An_{D(S)[x]}(u) = An_{D(S)[x]}(Bu) = D(S)[x]F$. Let put $v = e^G + Bu$ then v is an extension of u wiith the help of G in view of lemma 2.10, $An_{D(S)[x]}(v) = An_{D(S)[x]}(e^G + Bu) = An_{D(S)[x]}(e^G) An_{D(S)[x]}(Bu) = An_{D(S)[x]}(e^G) An_{D(S)[x]}(u) = D(S)[x]FG$. Then FG generates any segment of v

Proposition 2.12. Let F and G be monic polynomial and u is linear recurring sequence such that e^F be the extension of u with the help of G. If F and G are comaximales, then $L_S(S) = D(S)[x]u$ and $An_{D(S)[x]}(u) = D(S)[x]F$.

Proof: Let F and G be comaximales. There exists $H, K \in D(S)[x]$ such that $GK + FH = 1$. Then $e^F = HGe^F + KFe^F = HGe^F$. Since e^F be the extension of u with the help of $G, u = Ge^F$, then $e^F = Hu$. Hence by lemma 2.7, [3]; $D(S)[x]u = D(S)[x]e^F = L_S(F)$ and

$$An_{D(S)[x]}(u) = An_{D(S)[x]}(D(s)[x]u) = An_{D(S)[x]}(D(S)[x]e^F) = D(S)[x]e^F.$$

In the sequel, we give a new distinct improvement of Berlekamp - Massey algorithm on linear recurring sequences.

Let $d \in Z$ and note by $\varphi_d : Z \to R$ the function defined by:

$$\varphi_d(x) = \begin{cases} 1 \; if \; d \leq 0 \\ x^d \; if \; d \geq 0 \end{cases} (*)$$

by using the function φ defined below (*) , we have the following proposition .

Proposition 2.13. Let $r, s \in N$ and $H(x), G(x) \in S[x]$. Then $G(x) = \varphi_{r-s}(x)H(x) - \varphi_{s-r}(x)H(x)$ if and only if

$$G(x) = \begin{cases} H(x) - x^{s-r}H(x) if \; r \leq s \\ x^{r-s}H(x) - H(x) \; if \; r > s \end{cases}$$

In the sequel we improve the Berlekamp - Massey algorithm by changing in step (t+1)

$$G_{t+1}(x) = \begin{cases} G_t(x) - x^{(k_s)-(k_t)}u_t(k_t)u_s(k_s)^{-1}G_s(x) \quad if \quad k_t \leq k_s) \\ x^{k_t-k_s}G_t(x) - u_t(k_t)u_s(k_s)^{-1}G_s(x) \quad if \quad k_t > k_s) \end{cases}$$

with

$$G_{t+1}(x) = \varphi_{k_t-k_s}(x)G_t(x) - \varphi_{k_s-k_t}(x)u_t(k_t)u_s(k_s)^{-1}G_s(x)$$

with the help of function defined in (*)

By using the relation $G_{t+1}(x) = \varphi_{k_t-k_s}(x)G_t(x) - \varphi_{k_s-k_t}(x)u_t(k_t)u_s(k_s)^{-1}G_s(x)$ obtained with the help of function φ , the Berlecamp-Massey algorithm is the follows:

(1) We put on $G_0 = 1$, m_0 and $u_0\overline{(0, l - m_0 - 1)} = u_0\overline{(0, l - 1)}$ where $u_0 = G_0 u = u$. If $u_0\overline{(0, l - 1)} = 0$ then $l_u(G_0) > l$ and $G(x) = G_0(x), m_u(l) = m_0 = 0$.Otherwise $u_0 = (0, \dots, 0, u_0(k_0), \dots), u_0(k_0)1 \neq 0$, $k_0 = k_0(G) < l - m_0$

(2) We put on $G_1(x) = x^{k_0+1} - u_0(k_0 + 1)u_0(k_0)x^{k_0}G_0(x), m_1 = k_0 + 1$ and calculate $u_1\overline{(0, l - m_1 - 1)}$ where $u_1 = G_1 u$. If $u_1\overline{(0, l - m_1 - 1)} = 0$, then $l_u(G_1) > l$ and $G(x) = G_1(x)$, $m_u(l) = m_1 = 0$. Otherwise $u_1 = (0, 0, \dots, u_1(k_1), 0 \dots), u_1(k_1) \neq 0$, $k_1 = k_1(G) < l - m_1$

(3) Now let us suppose that we have built the polynomial $G_0, G_1, \dots, G_t$ let $k_j = k_u G_j$, $l_j = l_u(G_j) = k_j + m_j$ for $j \in o, t$

(4) We will define $s = \max\{i \in 0, t, m_i < m_t\}$ and put on

$$G_{t+1}(x) = \varphi_{k_t - k_s}(x)G_t(x) - \varphi_{k_s - k_t}(x)u_t(k_t)u_s(k_s)^{-1}G_s(x)$$

Where $u_j = Gu$ for $j \in 0, t$. Let $u_{t+1} = G_{t+1}u$. then $l_u(G_{t+1}) > l_u(G_t)$ and the following statement is true: either $u_{t+1}(0, l - m_{t+1} - 1) = 0$, and $G(x) = G_{t+1}(x), m_u(l) = m_{t+1}$, or $l_u(G_{t+1}(x)) = k_{t+1} + m_{t+1} = l_{t+1} < l$ and $G_{t+1}(x)$ is the polynomial of minimal degree which generates the segment $u\overline{(0, l_{t+1} - 1)}$.

Application:

(1) Let $u = (1,2,3,4,5 \dots)$ a sequence in Q^+ and l = 4. Let's find a monic polynomial which generates the segment $u\overline{(0, l - 1)}$.

- put on $G_0 = 1, m_0 = 0$, $u_0 = G_0 u = u$, $u_0\overline{(0,3)}$, (0,1,2,3) then $k_0 = 1 < 4 - 0 = 4$
- put on $G_0 = x^2 - u_0(2)u_0(1)^{-1}xG_0 = x^2 - 2x$, $m_1 = 2, u_1 = G_1 u$, $u_1\overline{(0,1)} = (0, -1)$. then $k_1 = 1 < 4 - 2 = 2$
- put on $G_2 = \varphi_{-1}(x)G_1 - \varphi_1(x)u_1(1)u_0(1)^{-1}G_0 = x^2 - 2x + 1$, $m_2 = 2, u_2 = G_2 u$, $u_2\overline{(0,1)} = (0,0)$. Then $G(x) = G_2(x) = x^2 - 2x + 1$ generates the segment

$u\overline{(0,3)}$. for all positive integer n, $G(x)u(n) = 0$, Then u is linear recurring sequence with minimal polynimial

(2) Let $u = (1,2,3,4,5, \ldots)$ a sequence in a semifield containing Q^+ as subsemifield and $l = 4$

- $G_0 = 1, m_0 = 0, u_0 = G_0u = u, u_0\overline{(0,3)}, (1,0,1,1)$ then $k_0 = 1 < 4 - 0 = 4$
- put on $G_1 = x - u_0(1)u_0(0)^{-1}G_0 = x, m_1 = 1, u_1 = G_1u, u_1(0,3) = (0,1,1)$. then $k_1 = 1 < 4 - 1 = 3$
- put on $G_2 = \varphi_1(x)G_1 - \varphi_{-1}(x)u_1(1)u_0(0)^{-1}G_0 = x^2 - 1, m_2 = 2, u_2 = G_2u$, $u_2\overline{(0,1)} = (0,1)$. Then $k_2 < 4 - 2 = 2$
- put on $G_3 = \varphi_0(x)G_2 - \varphi_0(x)u_2(1)u_1(1)^{-1}G_1 = x^2 - x + 1, m_3 = 2, u_3 = G_3u, u_3\overline{(0,1)} = (0,0)$. Then $G(x) = G_3(x) = x^2 - x + 1 \in Q^+[x]$ generates the segment $u(0,3)$ for all positive n, $G(x)u(n) = 0$ then u is linear recurring sequence with minimal polynomial $G(x)$.

(3) Let $u = (1,2,3,4, \ldots)$ a sequence in a semifield containing Q^+ as subsemifield and $l = 4$, let's find a monic polynomial which generates the segment $u(0, l - 1)$.

- $G_0 = 1, m_0 = 0, u_0 = G_0u = u, u_0\overline{(0,3)}, (1,2,3,4)$ then $k_0 = 1 < 4 - 1 = 4$

- $G_0 = x - u_0(1)u_0(0)^{-1}G_0 = x - 2, m_1 = 1, u_1 = G_1u, u_1\overline{(0,2)}, (0,-1,-2)$ then $k_1 = 1 < 4 - 1 = 3$
- put on $G_2 = \varphi_1(x)G_1 - \varphi_{-1}(x)u_1(1)u_0(1)^{-1}G_0 = x^2 - 2x + 2, m_2 = 2, u_2 = G_2u$, $u_2\overline{(0,1)} = (1,-6)$. Then $k_2 = 0 < 4 - 2 = 2$

- put on $G_3 = \varphi_{-1}(x)G_2 - \varphi_1(x)u_2(0)u_1(1)^{-1}G_1 = 2x^2 - 4x + 2, m_2 = 2, u_3 = G_3u$, $u_3\overline{(0,1)} = (0,0)$. Then $G(x) = G_3(x) = x^2 - 2x + 1$ generates the segment $u\overline{(0,3)}$. For all positive integer n, $G(x)u(n) = 0$. Then u is a linear recurring sequence with minimalpolynomial $G(x)$.

REFERENCES

1. J. S. Golan. Semirings and their applications, University of Haifa, Haifa, Israel. 1999.
2. V. Gupta and J. N. Chaudhari. On Partitioning Ideals of Semirings, Kungpook Math. J. 46(2006), 181-184.
3. L. Ngom, O. Diankha, D. Sow, Linear recurring sequences over zero-sum semirings, Springer Proceedings in Mathematics & Statistics 160, doi 10.1007/978-3-319-32902-4-18
4. A. A. Necheav and A. V. Mikhalev. Linear recurring sequences over modules. Actae Applicandae Mathematicae.42: 161-202. 1996.
5. A. A. Necheav. Linear recurring sequences over commutative rings. Discrete Math.Appl. Vol 2, No.6, pp. 659-683(1992) commutatifs. Comm. algebra 27 (1999), no. 12, 6175-6189.
6. E. M. Vechtomov and A. V. Cheraneva. Semi_elds and their properties, Journal of Mathematical Sciences, Vol.163, No. 6, 2009
7. J. N. Weinert Uber halbring and Halbkorper II. Acta. math. acad. hungar, 14(1963), 209-227.